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ANALYSIS OF THE ACCURACY OF SOLUTIONS OF THE TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEM

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UDC 536.2

The accuracy of solutions of the two-dimensional inverse heat-conduction problem is investigated. Exact and perturbed values of the temperature on the inner boundary are used as initial data.

Study of the nonstationary heating of structural elements, which are bodies of spherical and cylindrical shape and subjected to a high-temperature flux, requires knowledge of the external thermal loading conditions. The inverse problem, used to determine the conditions on the outer boundary according to the temperatures measured on the inner boundary, is examined below. In a number of cases it is necessary to take a two-dimensional heat-propagation model for bodies of spherical and cylindrical shape. For example, the two-dimensionality is taken into account for an intensive change in the free-stream flux parameters along the body generator and in the presence of anisotropy of the thermophysical properties [1]. A sufficiently large quantity of algorithms for solving inverse heat-transfer problems is known. Algorithms have been developed for solving inverse problems in linear and nonlinear formulations; algorithms taking into account structural changes in the material. These are mainly problems in a one-dimensional formulation which are justified in many cases of practical importance. For instance, if the installation of special heat-flux sensors is possible structurally in geometrically complicated spherical or cylindrical bodies, then in these cases there is no need to solve tedious multidimensional inverse problems. The determination of heat fluxes by using known heat-flux sensors is based on the solution of one-dimensional inverse heat-transfer problems. However, there exist few examples of practical investigations of the heat-transfer processes in constructions when the one-dimensional models do not adequately describe the actual physical processes and the installation of the above-mentioned heat-flux sensors is not possible. Spherical and cylindrical shells of small radius [2] are an example of such constructions. If there is a strict approach to the physical problem of heating, then a three-dimensional heat conductivity model is necessary to determine the external thermal boundary conditions for bodies of spherical and cylindrical shape. Unfortunately, a substantial growth of the calculations, resulting in large electronic computer time expenditure for the solution of the inverse problem, hinders the development of algorithms of inverse problems in a three-dimensional formulation. If the multidimensional nature of the heat conductivity in a cylindrical body is caused mainly by small radii of curvature, then the two-dimensional model describes the heat-conduction process well for a negligible heat-flux gradient along the cylinder generatrix. A two-dimensional heat-conduction model is also realized in the axisymmetric flow around a spherical body and the problem to determine the thermal boundary conditions can be solve in a polar coordinate system.

Let us consider a two-dimensional inverse boundary heat-conduction problem for a body of cylindrical shape. The heat flux $q_1(\varphi, \tau)$ is delivered to the outer surface, where τ is the time, φ is the angle of rotation in the cylindrical coordinate system 0, r, φ. As a result of the action of heat flux, a temperature field $T(r, \varphi, \tau)$ is realized in the body. We assume that the boundaries are heat insulated at $\varphi=0$, $\varphi=\varphi_k$ and $r = R_{ex}$. In this case the two-dimensional inverse heat-conduction problem is written as follows:

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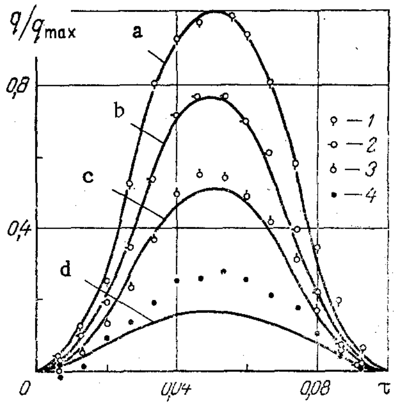

Fig. 1

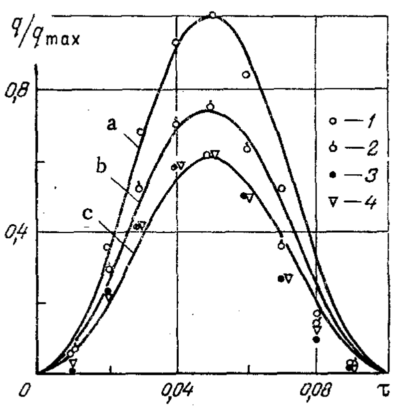

Fig. 2

Fig. 1. Restoration of the heat-flux density $q_1(\varphi, \tau)$, delivered to the external surface of a body for $\Delta Fo = 0.02$ (solid curves are the model heat flux, τ, in sec): a, 1) $\varphi = 90°$; b, 2) 60; c, 3) 30; d, 4) 0.

Fig. 2. Restoration of the heat-flux density with two maximums to the external body surface for $\Delta Fo = 0.02$ (the plane τ, 0, q; the solid curves are the model heat flux, τ in sec): a, 1) $\varphi = 45°$; b, 2) 18; c, 3) 0; 4) 90.

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial r}\left(\lambda(T)\frac{\partial T}{\partial r}\right) + \frac{\lambda(T)}{r}\frac{\partial T}{\partial r} + \frac{\partial}{\partial \varphi}\left(\lambda(T)\frac{\partial T}{\partial \varphi}\right), \tag{1}$$

$$R_{ex} < r < R,\ 0 < \varphi < \varphi_k,\ 0 < \tau \leqslant \tau_m,$$
$$T(r, \varphi, 0) = \psi(r, \varphi),\ R_{ex} \leqslant r \leqslant R,\ 0 \leqslant \varphi \leqslant \varphi_k, \tag{2}$$

where C(T) is the specific heat, and $\lambda(T)$ is the heat conduction.

The boundary conditions are

$$q_2(R_{ex}, \varphi, \tau) = \frac{\partial T(R_{ex}, \varphi, \tau)}{\partial r} = 0, \tag{3}$$

$$q_3(r, 0, \tau) = \frac{\partial T(r, 0, \tau)}{\partial \varphi} = 0, \tag{4}$$

$$q_4(r, \varphi_k, \tau) = \frac{\partial T(r, \varphi_k, \tau)}{\partial \varphi} = 0,\ 0 < \tau \leqslant \tau_m. \tag{5}$$

The temperature on the boundary $r = R_{ex}$

$$T(R_{ex}, \varphi, \tau) = T^*(\varphi, \tau) \simeq \xi(\varphi, \tau),\ 0 \leqslant \varphi \leqslant \varphi_k,\ 0 < \tau \leqslant \tau_m. \tag{6}$$

is the initial data for solution of the problem formulated. The sign $\simeq$ in (6) means that results of measuring the temperature $\xi(\varphi, \tau)$, obtained during a high-temperature experiment and which differ from the exact values of the temperature $T^*(\varphi, \tau)$ by the magnitude of a certain integral error δ_{L_2} are used as initial data. The desired heat-flux density delivered to the external surface of the cylindrical body is

$$q_1(\varphi, \tau) = -\lambda(T)\frac{\partial T(R, \varphi, \tau)}{\partial r},\ 0 \leqslant \varphi \leqslant \varphi_k,\ 0 < \tau \leqslant \tau_m. \tag{7}$$

Let us examine one of the particular cases when

$$\lambda(T)/C(T) = a = \text{const}. \tag{8}$$

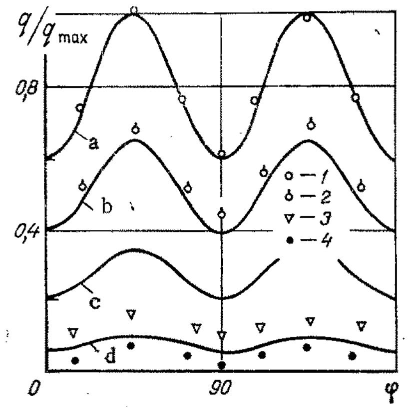

Fig. 3

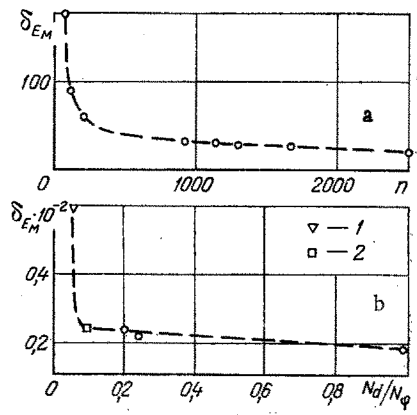

Fig. 4

Fig. 3. Restoration of the heat-flux density with two maximums to the external body surface for $\Delta Fo = 0.02$ (the plane φ, 0, q, solid curves are the model heat flux, and φ in degrees): a, 1) $\tau = 0.05$ sec; b, 2) 0.03; c, 3) 0.08; d, 4) 0.01.

Fig. 4. Dependence of error in the solutions of the two-dimensional problem δ_{E_m}, kW/m^2, on the number of difference mesh nodes $n=n_\varphi \times n_r$ (a) and on the number of temperature measurement points on the body interior surface N_d/N_φ (b): 1) $n_d = 3$; 2) 5.

A formulation of problem (1)-(6) under condition (8) is proposed in [2]. A description is given there for the algorithm solving the two-dimensional inverse heat-conduction problem which is based on an extremal formulation using the method of conjugate gradients. The method of variable directions is here used for the calculations. Moreover, the specific generality of the method permits making the formulation of two-dimensional inverse problems complicated, for instance, by inserting different nonlinearities. Individual results of solving the two-dimensional heat-conduction problem by using exact and perturbed initial data are presented in [3]. As a rule the algorithms of inverse problems are investigated by using special methodological examples. Since the problem under consideration is solved by a numerical method, then it would be natural to compare them with analytic solutions of special methodological problems to determine the accuracy of the solutions obtained. But it is sufficiently difficult to select an example with an analytic solution for inverse problems in a two-dimensional formula. Consequently, the accuracy of the solution of this inverse problem was investigated by using special numerical experiments. Model thermal fluxes were used, whose laws of variation in space and time were close to the real thermal loading. For such model heat fluxes, the temperatures on the external surface were determined by using known methods. Consequently, a complex of initial data for the solution of the two-dimensional inverse heat-conduction problem and control values of the desired solutions were obtained. During the investigation it would be necessary to clarify what laws of variation of the heat-flux density in space and time can be restored by using the solution of the two-dimensional inverse heat-conduction problem with acceptable accuracy for practice. Parameters of the difference mesh and the least number of fixed points with known temperature needed on the internal body boundary for which the necessary accuracy in the solution of the inverse problem is achieved were estimated from this experiment. This latter is of interest from the practical viewpoint. Since a two-dimensional inverse problem is solved, then it is therefore necessary to have initial data distributed not only in time but in the space coordinate as well. If the temperature is measured by using a thermocouple, say, then the quantity of measurements in the space coordinate will be directly related to the quantity of thermocouples installed on the body interior surface. It can be said that the quantity of thermocouples should not be less than three since only in this case is the information about the temperature of both the inner and the side boundaries of the body taken into account. A definite answer to the question of the quantity of temperature measurement points can be given just by a specially formulated numerical experiment.

The fact that experimental values of the temperature were used in determining the thermal fluxes by using inverse heat-conduction problems was indeed taken into account in the experimental investigation performed. It is known that the temperature measured during the experiment is under definite assumptions, the sum of useful information and different errors associated with the imperfections of the measuring and recording apparatus. Hence, the perturbed values of the temperature in both space and time are ordinarily used as initial data in the solution of two-dimensional inverse-conduction problems. In practice, the signal being recorded in an analysis of experimental information is represented in the form of the sum of the useful signal and a random component. The random component was generated by a pseudorandom number sensor. Two random number distribution laws were utilized here, uniform and normal.

Analysis of the results of an experiment using exact and perturbed initial data to solve the two-dimensional inverse heat-conduction problem showed that the method of determining the heat-flux density delivered to the external surface of a cylindrical body by means of the results of measuring the temperature on the internal heat insulated surface yields good results for sufficiently complicated thermal loading laws. A space—time thermal loading law (coordinate system 0, τ, φ, q) is displayed graphically in Fig. 1 on the external boundary of a cylindrical body in the τ, 0, q plane. The law of delivered heat-flux density variation has an analogous representation in the φ, 0, q plane. A more complex dependence of the heat flux is shown in Figs. 2 and 3. Here, the heat-flux density in the plane φ, 0, q already has two maximal values. In this latter case it would be necessary to find the solution of a two-dimensional inverse heat-conduction problem that is a dependence describing the low-frequency oscillatory process. Despite the perturbation of the initial data, which reaches 5% of the maximal value of the temperature, the method under investigation permitted restoration of the model heat-flux value with sufficiently high accuracy.

Results on an investigation of the influence of the difference mesh parameters of a finite-difference method used in solving the inverse problem are of interest. An inverse two-dimensional heat-conduction problem was solved on a difference mesh with a different quantity of nodes suring the experiment. The solutions obtained were compared with the model heat flux. The accuracy of the solutions was estimated according to the errors

$$\delta_{E_m} = \sqrt{\sum \sigma^2_{qi,j}}, \quad \begin{matrix} i = 1,\ 2,\ \ldots,\ m, \\ j = 1,\ 2,\ \ldots,\ k, \end{matrix}$$

where $\sigma^2_{q_{i,j}}$ is the variance of the solutions inverse problem for the model heat flux.

The fundamental results are represented in Fig. 4. As is seen from the figure, two-dimensional inverse problems in the formulation under consideration should be solved on a difference mesh with n = 800-1000 nodes. A further increase in the quantity of nodes does not result in a substantial increase in the accuracy of the inverse problem solutions obtained.

It was noted above that the question of the quantity of thermocouples installed to measure the temperature on the body internal surface is important in a computational experimental determination of the deliverable heat-flux density by the methodology using the solution of a two-dimensional inverse heat-conduction problem. In this connection, the dependence of the accuracy of the solution of the two-dimensional inverse heat-conduction problem on the quantity of points with known temperature on the body interior boundary must be determined. The results of the investigation performed showed that the quantity of temperature measurement points needed from the viewpoint of the accuracy of solving the two-dimensional inverse heat-conduction problem fluctuates from $n_d = 3$ to $n_d = 9$ (Fig. 4). It should be noted that the quantity of measurement points $n_d = 5$ turns out to be more acceptable for the majority of considered modifications of the solution of two-dimensional problems.

NOTATION

q, heat-flux density; 0, r, φ , polar coordinate system; R, R_{ex}, external and internal surface radii for the cylindrical body; φ_k , greatest value of the variable τ, time; τ_m, greatest value of the variable τ; T, temperature; C, coefficient of volume specific heat; λ, heat-conduction coefficient; a, thermal diffusivity coefficient; ξ, temperature measured in experiment; n, number of difference mesh nodes; n_d, number of temperature measurement points; ΔFo, Fourier number step; δ_{E_m}, integrated error; and σ^2, variance.

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OPTIMAL PLANNING OF MEASUREMENTS IN NUMERICAL EXPERIMENT DETERMINATION OF THE CHARACTERISTICS OF A HEAT FLUX

E. A. Artyukhin and S. A. Budnik

UDC 536.24

The authors present an algorithm and analyze results of optimization of a temperature measurement scheme for solving inverse heat-conduction boundary problems.

In experimental investigations and the development of thermal regimes for various thermally loaded engineering items, there has recently been wide use of methods of diagnosing heat fluxes based on solving inverse heat-conduction boundary problems (IBP) [1]. The use of these methods requires careful analysis of the computing properties of the IBP solution algorithm (e.g., rate of convergence, stability, errors in recovering the desired functions) and determining the conditions for conducting the temperature measurements to achieve maximum reliability of results of the diagnosis.

The mathematical modeling data show that the accuracy of recovering the boundary thermal conditions can be increased by choosing the location of the thermal sensors in the test body, and also by solving the IBP in a redefined formulation [2]. Here the question arises of the baseline choice of the number of thermal sensors and their rational location in the specimen. The present paper analyzes this problem from the standpoint of theory of an optimal experiment [3, 4].

We consider a planar unbounded plate of thickness b in which the heat-transfer process is described by the following equation of unsteady heat conduction with boundary conditions of the second kind:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ 0<x<b,\ 0<\tau\leqslant\tau_m, \tag{1}$$

$$T(x,\ 0) = T_0(x),\ 0\leqslant x\leqslant b, \tag{2}$$

$$-\lambda(T(0,\ \tau))\frac{\partial T(0,\ \tau)}{\partial x} = q_1(\tau), \tag{3}$$

$$-\lambda(T(b,\ \tau))\frac{\partial T(b,\ \tau)}{\partial x} = q_2(\tau). \tag{4}$$

The IBP consists of defining the heat-flux density on one of the boundaries, e.g., $q_1(\tau)$, or simultaneously on both boundaries, $q_1(\tau)$ and $q_2(\tau)$, using the mathematical model of Eqs. (1)-(4) and the measured temperature data at a certain limited number N of points of the plate with coordinates $x=X_i$, $i=\overline{1,\ N}$:

$$T^{\exp}(X_i,\ \tau) = f_i(\tau),\ i=\overline{1,\ N}. \tag{5}$$

Efficient iterative computing algorithms for recovering the above characteristics have been proposed, for example, in [1, 2], in which the approximate solution of the inverse problem is determined from the uncoupling condition:

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